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SHORTCUTS, DIVERSIONS, AND MAXIMAL CHAINS IN PARTIALLY ORDERED SETS

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Abstract. An algorithm is described for finding the maximal weight chain between two points in a locally finite partial order under the restriction that all but κ (or fewer) successive pairs in the chain belong to a given subset of the partial order relation. Applications of the method in molecular genetics, critical path scheduling, and other fields are discussed.

Let P be a set with two locally finite partial orderings, called *strong* and *weak*, in which ordered pairs are designated by $p \ll q$ and $p < q$, respectively. The strongly ordered pairs constitute a subset of the weakly ordered pairs, or, in other words, if $(p, q) \in P \times P$ and $p \ll q$, then $p < q$. Let W be any non-negative, real-valued function on P, which we will call the *weight function*. A *chain* between $p \in P$ and $q \in P$ is any sequence

$$p = p_1 < p_2 < ... < p_\lambda = q$$

and the *weight* of this chain is defined to be $\Sigma_{i=1}^{\lambda} W(p_i)$. For κ a non-negative integer, a κ*-weak chain* is a chain in which all but κ (or fewer) pairs of successive terms are strongly ordered. A κ-weak chain is thus $(\kappa+1)$-weak, $(\kappa+2)$-weak, etc. A *zero-weak chain* is a chain in which all pairs of successive terms are strongly ordered.

The problem of finding a κ-weak chain of maximal weight between two elements p and q arises in such diverse fields as molecular genetics, critical path scheduling, bipartite graph theory, and traffic routing. We shall discuss one example from each of these. The method to be described is based on the association, to each weight function W on P, a family of incidence functions V_κ, whose values are the maximal chain weights for κ-weak chains between two points in P. These may be added to the collection of such functions which are already known to be useful in the study of partially ordered sets. In [3], Rota has drawn attention to the importance of incidence functions in combinatorial theory.

Lemma. *Let $\leqslant$ and $<$ be a strong and a weak partial order, respectively, on the same set P. For κ a non-negative integer and $(p, q) \in P \times P$, there exists a κ-weak chain between p and q if and only if at least one of the following holds:*

(i) $p = q$;

(ii) $p \leqslant q$ *and* $\kappa = 0$;

(iii) $p < q$ *and* $\kappa > 0$.

Proof. In (i), the chain consisting of p alone contains no successive pairs and hence it is zero-weak. Then it is κ-weak for all $\kappa \geq 0$. In (ii), the chain consisting of p and q is zero-weak, and in (iii), it is 1-weak, and hence κ-weak for $\kappa \geq 1$.

To show that the existence of chains implies the necessity of at least one of (i), (ii) or (iii), assume $p \neq q$. For any zero-weak chain between p and q, the transitivity of $\leqslant$ implies $p \leqslant q$. For any κ-weak chain ($\kappa > 0$), the transitivity of $<$ implies $p < q$.

In the following theorem, we use dynamic programming to associate, to each weight W, an incidence function $V_\kappa(p, q)$ measuring the maximal κ-weak chain weight between p and q. If no chain exists, we say $V_\kappa(p, q) = -\infty$.

Theorem. *Let P, $\leqslant$ and $<$ be as in the lemma. Let W be any real-valued, non-negative weight on P, and for $p < q$ let $V_\kappa(p, q)$ be the maximal weight for κ-weak chains between p and q. Then*

(i) $V_\kappa(p, q) = W(q)$, *if* $p = q$;

(ii) $V_\kappa(p, q) = W(q) + \max_{s \leqslant q} V_\kappa(p, s)$, *if* $p \leqslant q$ *and* $\kappa = 0$;

(iii) $V_\kappa(p, q) = W(q) + \max_{s \leqslant q, r < q} \{V_\kappa(p, s),\ V_{\kappa-1}(p, r)\}$, *if* $p < q$ *and* $\kappa > 0$.

Proof. By the lemma, all chains must satisfy the conditions on p, q, and κ specified in at least one of (i), (ii), or (iii). In Case (i), the theorem follows directly from the definition of the weight on a chain.

In Case (ii), we proceed by induction, using the local finiteness of P. Suppose the theorem is true for $\kappa = 0$ and all pairs (p, s), where $s \leqslant q$. Then since all zero-weak chains are of form $p \leqslant \ldots \leqslant s \leqslant q$, we write

$$\begin{aligned} V_0(p,q) &= \max_{\{\text{chains } p \leqslant \ldots \leqslant s \leqslant q\}} \{W(p) + \ldots + W(s) + W(q)\} \\ &= \max_{s \leqslant q} \left\{ \max_{\{\text{chains } p \leqslant \ldots \leqslant s\}} \{W(p) + \ldots + W(s) + W(q)\} \right\} \\ &= W(q) + \max_{s \leqslant q} \left\{ \max_{\{\text{chains } p \leqslant \ldots \leqslant s\}} \{W(p) + \ldots + W(s)\} \right\} \\ &= W(q) + \max_{s \leqslant q} V_0(p,s). \end{aligned}$$

In Case (iii), the induction is on both q and κ. Suppose the theorem is true for $\kappa-1, \kappa-2, \ldots$ for all pairs (p, r) where $r < q$; and true for κ, for all pairs (p, s) where $s \leqslant q$. Then since a κ-weak chain is either of the form $p < \ldots < s \leqslant q$, where $p < \ldots < s$ is κ-weak; or of the form $p < \ldots < r < q$, where $p < \ldots < r$ is at most $(\kappa-1)$-weak, we may write

$$\begin{aligned} V_\kappa(p,q) &= \max_{\{\kappa\text{-weak chains } p < \ldots < q\}} \{W(p) + \ldots + W(q)\} \\ &= W(q) + \max \left\{ \max_{s \leqslant q} \left\{ \max_{\{\kappa\text{-weak chains } p < \ldots < s\}} \{W(p) + \ldots + W(s)\} \right\}, \right. \\ &\qquad \left. \max_{r < q} \left\{ \max_{\{(\kappa-1)\text{-weak chains } p < \ldots < r\}} \{W(p) + \ldots + W(r)\} \right\} \right\} \\ &= W(q) + \max_{s \leqslant q, r < q} \{V_\kappa(p,s), V_{\kappa-1}(p,r)\}. \end{aligned}$$

The initial step in the induction proofs of Cases (ii) and (iii) is provided by Case (i).

Remark. For P finite, the maximal weight for a κ-weak chain anywhere in P is $\max_{(p,q)\in P\times P} V_\kappa(p,q)$ and in some applications this is the quantity of interest.

After the incidence function V_κ is evaluated, the next step is to construct a κ-weak chain between p and q having maximal weight. Rather than examine all possible chains, a task which rapidly becomes impractical as the number of elements between p and q increases, the aim is to find an algorithm for identifying a maximal chain with reasonable effort and time. This may be achieved by using the functions V_κ and the principle of consistent enumeration [1; Theorem 4, p. 40]. The latter asserts that the elements of a finite partial order, say $(P, <)$, may be totally ordered, say as $(P, \prec)$, such that for $(p, q) \in P \times P$, $p < q \Rightarrow p \prec q$.

Let n be the number of elements in P between p and q ($n < \infty$ by local finiteness), and m be the maximum number of elements *immediately preceding* any of these elements; that is

$$m = \max_{p<s\leq q} \{\text{number of } t\in P \text{ such that } t<s, \text{ but for no } r\in P \text{ does } t<r<s\}.$$

Then starting with the total order $\prec$, we can find a maximal κ-weak chain between p and q, including the calculation of the incidence function V_κ, using no more than Cmn computational steps, C being independent of m and n. First, as it may easily be shown, instead of examining all $s \leqslant q$ and all $r < q$ to calculate the incidence function in Cases (ii) and (iii) of the theorem, it suffices to find the maximum to examine just those s and r which immediately precede q. There are at most m of these, and they all precede q in the total order $\prec$. Thus by calculating the $V_\kappa(p, r)$ for r in the order specified by $\prec$, the procedure is well-defined recursively and will require a number of computational steps proportional to κmn, at most, to calculate V_κ.

Now consider the (at most) m elements immediately preceding q in the weak order. By the theorem, there is among them a $q_1 \leqslant q$, such that $V_\kappa(p, q_1) = V_\kappa(p, q) - W(q)$, or else a $q_1 < q$ such that $V_{\kappa-1}(p, q_1) = V_\kappa(p, q) - W(q)$. To find this element requires at most $2m$ search steps. The search procedure is repeated to find a suitable q_2 among the elements immediately preceding q_1, and so on. The definition of V_κ and the theorem assure us that we will eventually arrive at Case (ii) and then Case (i). The sequence $p < ... < q_2 < q_1 < q$ so obtained will be a maximal chain. Since this will contain n or less elements, the construction will involve at most $2mn$ steps. Thus the total of calculation and search steps will be proportional to mn.

Note that this construction produces only *complete chains*, i.e., chains in which p_i immediately precedes p_{i+1}, for $p = p_1 < ... < p_\lambda = q$. By the non-negativity of W, and local finiteness, any maximal chain can be extended to a complete chain of the same weight (which is thus also maximal).

Example 1. The prototype problem is how to find the *longest* κ-weak chain $p_1 < ... < p_\lambda$ in a finite P, where by longest we mean the highest possible value for λ. This is solved by assigning the uniform weight function $W \equiv 1$, and then calculating $\max_{(p,q) \in P \times P} V_\kappa(p, q)$.

Example 2. In the study of evolution at the molecular level, it is often necessary to "match up" two finite sequences according to certain criteria. Let $a_1, ..., a_m$ and $b_1, ..., b_n$ be two sequences, where all $a_i \in S$, $b_j \in S$ for some finite alphabet S. Let W be a weight function on $S \times S$. We are required to construct a sequence of pairs (a_{i_1}, b_{j_1}), $(a_{i_2}, b_{j_2}), ..., (a_{i_\lambda}, b_{j_\lambda})$ such that

$$(*) \qquad \mu < \nu \Rightarrow i_\mu < i_\nu \text{ and } j_\mu < j_\nu$$

and such that $\Sigma_{\gamma=1}^{\lambda} W(a_{i_\gamma}, b_{j_\gamma})$ is maximal, subject to the constraint that

$$(**) \qquad i_{\nu+1} - i_\nu = j_{\nu+1} - j_\nu = 1$$

for all but κ or fewer values of ν. One such weight function favoring pairs (a, b), where $a = b$, was introduced in [2]. The condition (*) provides a partial order on the set of all pairs (a_i, b_j) and condition (**) provides a stronger partial order on these pairs. The functions V_κ were introduced in [4] to find maximal κ-weak chains for this problem.

Example 3. In the planning of complex projects, use is made of a partial order relating the component tasks of the project, where $p \leqslant q$ if task p must be completed before task q may be started. The weight function $W(p)$ specifies the time necessary to perform task p. Then the maximal chain through the partial order, the *critical path*, may be found using the function V_0. This is the path on which individual task completion schedules must be most carefully followed to ensure the completion of the entire project on time. In more careful planning, we might wish to introduce a weak partial order which relates two tasks $p < q$ if p *might* have to be completed before q starts, due to unforseen shortages of labour, material, plant space etc. Maximal κ-weak chains are sub-critical paths which take into account the possibility of at most κ of the contingencies represented by the partial order $<$.

Example 4. Consider a bipartite graph B consisting of two rows of vertices $a_1, ..., a_m$ and $b_1, ..., b_n$, together with (straight) edges joining some of the pairs (a_i, b_j). The problem is to find a maximal planar subgraph of B, that is a subgraph with the largest possible number of edges such that no two edges intersect. This is solved by introducing a weak partial order $<$, where $(a_i, b_j) < (a_k, b_l)$ if $k-i$ and $l-j$ are both non-negative. The weight function is $W(a_i, b_j) = 1$, if there is and edge between a_i and b_j, and zero elsewhere. Then we can find a maximal planar subgraph of B by using V_0. If we wish to find the maximal planar subgraph with at most $\kappa+1$ mutually disjoint, connected components, we can use our procedure if we first introduce a strong relation $\sim$, where $(a_i, b_j) \sim (a_k, b_l)$ only if $(a_i, b_j) < (a_k, b_l)$ and $k = i$ or $l = j$. This relation is not a partial order, since it is not transitive, so we define $\ll$ to be the smallest partial order containing $\sim$, and then search for a κ-weak chain. The fact that our method only produces complete chains ensures that any subchain in which all successive pairs are strongly ordered will represent a connected subgraph.

Example 5. The examples presented above can all be thought of as producing κ-weak chains by introducing "diversions" in zero-weak chains. The theorem and algorithm can also be adapted to finding "shortcuts" in zero-weak chains. This requires replacing "max" by "min" in the algorithm and theorem, restricting attention to complete chains, and changing our convention so that the non-existence of chains between p and q implies $V_\kappa(p, q) = +\infty$. Thus, in analogy to Example 1 above, we might want to find the *shortest* complete κ-weak chain between two extreme points in a finite partial order. The problem of finding the best route between two points on a map might also be abstracted in terms of minimal chains through partial orders. The points in P could represent road sectors, cities, depots, etc; the weights might represent distances, travel time, toll charges, or layover costs; and successive points which are not strongly ordered could represent shortcuts over second-class or poorly serviced roads, or use of alternative means of transportation involving reloading.

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